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NEO-KEYNESIAN AND NEO-CLASSICAL MACROECONOMIC MODELS: STABILITY AND LYAPUNOV EXPONENTS

Abstract
The non-linear approach to economic dynamics enables us to study traditional economic models using modified formulations and different methods of solution. In this article we compare the dynamic properties of the Keynesian and Classical macroeconomic models. We start with an extended dynamic IS-LM neoclassical model generating the behavior of the real product, the interest rate, expected inflation, and the price level over time. Limiting behavior, stability, and the existence of limit cycles and other specific features of these models will be compared.

Keywords: macroeconomic models, Keynesian and Classical model, non-linear differential equations, linearization, asymptotical stability, Lyapunov exponents

JEL classification: C00, E12, E13

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1. Macroeconomic Models

In this article we try to revive traditional models based on the IS-LM structure. Such models are different from the models which utilize the micro-foundations of macroeconomic theory or rational expectations and nowadays prevail in modern analysis, but they are still the subject of analysis in many professional journals and books.[1] We provide a non-linear reformulation of models of IS-LM structure to better comprehend the nature of the economy, which contradicts linear principles. In this way we get non-linear models and try to analyze them with the help of appropriate methods.

For the non-linear model presented here we found inspiration in (Chiarella et al., 2000). This book introduces the IS-LM-PC model. PC denotes that the IS-LM model is augmented by price-wage dynamics, i.e., by the modified Phillips curve, including inflation expectations. We develop this model in the following way. We replace the price-

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[1] In (Turnovsky, 2000) we can find not only models of traditional macro-dynamics, but also models of inter-temporal optimization and rational expectations models. The last two represent the majority approach to the modern analysis of economic systems.

-wage dynamics by price-marginal cost (PMC) dynamics. The modified model will be denoted IS-LM-PMC.

The IS-LM-PMC model is structured as four differential equations. The first equation describes the commodity market, the second describes the money market, and the third describes the relationship between marginal cost and prices. The fourth equation deals with inflation expectations. We assume adaptive expectations. The left-hand side of the commodity market equation (1) contains the gap between demand (investment) and supply (savings) in the aggregate commodity market. The left-hand side of equation (2) contains the gap between money supply and money demand. And the left-hand side of equation (3) contains the gap between the price level and marginal cost. Notice that from the general point of view the IS-LM-PMC structure could be common to both the Keynesian and neoclassical approaches. The difference lies only in the style of imputation of the equalizing factors of the model. The Keynesian approach states that change in production equalizes the commodity market (IS), change in the interest rate equalizes the money market (LM), and change in the price level equalizes the price level and marginal costs. The neoclassical approach assumes that change in the interest rate equalizes the commodity market (IS), change in the price equalizes the money market (LM), and change in production equalizes the price level and marginal costs (PMC). This paper aims to analyze the consequences of the Keynesian and neoclassical approaches to the IS-LM-PMC structure for the dynamics of the related models.

We begin with the description of the Keynesian IS-LM-PMC model. Let (in continuous time $t \ge 0$) $Y(t)$, $S(\cdot,\cdot)$ and $I(\cdot,\cdot)$ denote, respectively, the real product, savings, and real investments of the considered economy. Recall that for the nominal interest $R(t)$ it holds that $R(t) = r(t) + \pi^e(t)$, where $r(t)$ is the real rate of interest and $\pi^e(t)$ is expected inflation, in contrast to inflation $\pi(t)$. The dynamics of the IS model are then given by the following differential equation – see e.g. (Takayama, 1994)

$$\dot{Y} = \alpha\left\{I(Y(t), r(t)) - S(Y(t), r(t))\right\}$$

or, on taking logarithms, by

$$\frac{dy(t)}{dt} = \alpha\left\{i(y(t), r(t)) - s(y(t), r(t))\right\} \tag{1}$$

where $y(t) = \ln Y(t)$, and $i(\cdot,\cdot) = \dfrac{I(\cdot,\cdot)}{Y(\cdot,\cdot)}$ and $s(\cdot,\cdot) = \dfrac{S(\cdot,\cdot)}{Y(\cdot,\cdot)}$ are, respectively, the propensity to invest and the propensity to save. Observe that for an equilibrium point $Y(t) \equiv Y^\star$, $y(t) \equiv y^\star$, $r(t) \equiv r^\star$, we have $I(Y^\star, r^\star) = S(Y^\star, r^\star)$ or $i(y^\star, r^\star) = s(y^\star, r^\star)$.

Denoting by $p(t)$ the price level at time *t*, the dynamics of the money market are described by the following differential equation

$$\frac{dr(t)}{dt} = \beta\left\{\ell(y(t), R(t)) - \ln\frac{M^s}{p(t)}\right\} = \beta\left\{\ell(y(t), r(t) + \pi^e(t)) - (m^s - \overline{p}(t))\right\} \tag{2}$$

where $\ell(y(t), R(t)) = \ln(L(Y(t), R(t)))$; $m^s = \ln M^s$; $\overline{p}(t) = \ln p(t)$; and $L(\cdot,\cdot)$ and

M^s are reserved for demand for money and the money supply respectively. In (1) and (2), α and β are positive constants signifying the speed of adjustment of the respective market.

To obtain a complete dynamic model of the economy we need to include equations for expected inflation $\pi^e(t)$ and the price level $p(t)$. According to (Tobin, 1975), for $\pi^e(t)$ the following adaptive equation is valid

$$\frac{d\pi^e(t)}{dt} = \gamma\left[\pi(t) - \pi^e(t)\right] \tag{3}$$

where γ is the coefficient of adaptation and $\pi(t)$ is inflation. Recalling that $\pi(t) = \frac{\dot{p}(t)}{p(t)} = $ $= \frac{d}{dt}\overline{p}(t)$, from (3) we immediately get

$$\frac{d\pi^e(t)}{dt} = \gamma\left[\frac{d}{dt}\overline{p}(t) - \pi^e(t)\right] \tag{4}$$

For what follows we need to express $\frac{d}{dt}\overline{p}(t)$ To this end we assume that the development of the price level $p(t)$ over time is in accordance with changes in the so-called cost function $C(y(t))$. In particular, the well-known condition of profit maximization $p(t) - \frac{dC(y)}{dy} = 0$ is the basis for the following adjustment formula for $p(t)$, where δ is a constant:

$$\frac{d\overline{p}(t)}{dt} = \delta\left(\frac{dC(y)}{dy} - e^{\overline{p}(t)}\right) \tag{5}$$

In fact, the above formula is in accordance with the traditional theory of perfectly competitive firms (see e.g. (Laider, Estrin, 1989)) and as such is interpreted in many treatises on monetary and price dynamics (cf. e.g. (Flaschel, Franke, Semmler, 1997)).

In what follows we shall use shorthand notations only, i.e., we replace $\frac{d\overline{p}(t)}{dt}$ by $\dot{\overline{p}}$, and do likewise for the time derivatives $\dot{y}$, $\dot{r}$, π^e, and $\frac{dC(y)}{dy}$ is replaced by $C'(y)$ Moreover, we shall often omit the argument *t*. Hence, (cf. (1), (2), (4), and (5)) using such a model the system describing an economy from the Keynesian point of view has the following form:

$$\left.\begin{aligned} \dot{y} &= \alpha[i(y,r) - s(y,r)] \\ \dot{r} &= \beta[\ell(y, r+\pi^e) - (m^s - \overline{p})] \\ \dot{\pi}^e &= \gamma[\dot{\overline{p}} - \pi^e] \\ \dot{\overline{p}} &= \delta[C'(y) - e^{\overline{p}}] \end{aligned}\right\} \tag{6}$$

where $i(y,r)$, $s(y,r)$, $\ell(y,r+\pi^e)$ and $C(y)$ are, respectively, the real investment, real savings, real money demand, and cost functions, depending on production *y*, the rate of interest *r*, (expected) inflation π^e, and the price level *p*.

Classical models that describe (commodity) price level, interest rate, production and expected inflation dynamics have a similar structure on the right-hand sides (RHS) of their differential equations, but the left-hand sides (LHS) are permuted as follows:

$$\left.\begin{aligned} \dot{r} &= \alpha[i(y,r)-s(y,r)] \\ -\dot{\overline{p}} &= \beta[\ell(y,r+\pi^e)-(m^s-\overline{p})] \\ \dot{\pi}^e &= \gamma[\dot{\overline{p}}-\pi^e] \end{aligned}\right\} \qquad (7)$$

Since for classical models the real product $y(t)$ is assumed to be constant, in (7) we ignore the equation $-\dot{y}=\delta[C^{'}(y)-e^{\overline{p}}]$.

The models just introduced form the basis for the establishment of macroeconomic models of price and monetary dynamics. Recall that the vector $\boldsymbol{x}^*=(y^*,r^*,\pi^{e*},\overline{p}^*)$, whose elements are obtained as a solution of the following set of equations:

$$\left.\begin{aligned} i(y,r) &= s(y,r) \\ \ell(y,r+\pi^e) &= m^s-\overline{p} \\ e^{\overline{p}} &= C^{'}(y) \end{aligned}\right\} \qquad (8)$$

is the equilibrium point of both the Keynesian model given by the set of equations (6) and the Classical model given by the set of equations (7). This equilibrium point is said to be (asymptotically) locally stable if every solution of the considered system starting sufficiently close to $\boldsymbol{x}^*$ converges to $\boldsymbol{x}^*$ as $t\to\infty$. Similarly, $\boldsymbol{x}^*$ is said to be (asymptotically) globally stable if every solution regardless of the starting point converges to $\boldsymbol{x}^*$. It is well known (cf. e.g. (Guckenheimer, Holmes, 1986) or (Takayama, 1994)) that an equilibrium point (and also a stable point) of the system need not exist, hence the system is unstable. Recall that having found the equilibrium points, the system need not converge to some or any of the equilibrium points (in the latter case the system is unstable). Furthermore, if the considered system is unstable and non--linear, then the system can also exhibit limit cycles (i.e., its trajectory remains in a bounded region) or even chaotic behavior. In other words, in contrast to the above phenomena, stability is equivalent to monotone or oscillating convergence toward the equilibrium point.

To identify chaotic behavior of a macroeconomic model, it is plausible to compare the dynamic behavior of the macroeconomic model with the exponential divergence of nearby trajectories measured by the so-called Lyapunov exponents. The most important of these is the maximal Lyapunov exponent, which is negative for stable models, positive for unstable models, and infinite for chaotic behavior – for details see (Lorenz, 1993).

2. Approximation and Linearization of the Models

To find an analytical form of output $y(t) = \ln Y(t)$, the interest rate $r(t)$, expected inflation $\pi^e(t)$ and the price level $p(t)$ we need to assume that the functions $i(\cdot,\cdot)$, $s(\cdot,\cdot)$, $C(\cdot)$ are of a specific analytical form. As usual, the functions $s(\cdot,\cdot)$, as well as demand for money $\ell(y,R)$, can be well approximated by linear functions, whereas it is necessary to approximate $i(\cdot,\cdot)$ and sometimes also $C(\cdot)$ by suitable non--linear functions. In what follows, we assume that savings $S(Y(t),r(t))$ can be well approximated by the following expression

$$S(Y(t),r(t)) = Y(t)\cdot[s_0 + s_1\cdot y(t) + s_2\cdot r(t)] \quad \text{with} \quad s_0 < 0, \, and \; s_1, s_2 > 0 \qquad (9)$$

Hence the propensity to save $s(\cdot,\cdot) = S(\cdot,\cdot)/Y(\cdot)$ can be written as

$$s(Y(t),r(t))def = \overline{s}(y(t),r(t)) = s_0 + s_1\cdot y(t) + s_2\cdot r(t) \qquad (10)$$

Similarly, the demand for money is described by the traditional Keynesian demand-for-money function in the following form

$$\ell(y(t),R(t)) = \ell_0 + \ell_1 y(t) - \ell_2 R(t) - \ell_3\pi^e(t) = \ell_0 + \ell_1 y(t) - \ell_2[r(t)+\pi^e(t)] - \ell_3\pi^e(t) \qquad (11)$$

where the parameters $\ell_i > 0,\ i = 0,1,2,3$ are given. On the other hand, it is convenient to assume that the propensity to invest $i(y(t),r(t))$ is a product of $\frac{1}{r(t)+1}$ and the so-called logistic function. Hence the propensity to invest is assumed to be given analytically as

$$i(y(t),r(t)) = \frac{1}{r(t)+1}\cdot\frac{k}{1+be^{-ay(t)}} \qquad (12)$$

where the parameters $k, a > 0$ and b is an arbitrary real number. Similarly, we shall assume that the cost function $C(\cdot)$ is also a logistic function given analytically as

$$C(y(t)) = \frac{h}{1+de^{-cy(t)}} \qquad (13)$$

where the parameters $h, c > 0$ and d is an arbitrary real number. Hence

$$\frac{dC(y)}{dy} = \frac{cdh}{(1+de^{-cy})^2}e^{-cy} \qquad (14)$$

and we can assume that the "central" part of $C(y(t))$ can be well approximated by a linear function

$$C(y(t)) = d_0 + d_1 y(t) \qquad (15)$$

Since $\pi^{e\star} = 0$, to calculate the values $y^\star, r^\star, p^\star$, on inserting (10), (11), (12) and (13) into (8) we have

$$\frac{1}{r^\star + 1} \cdot \frac{k}{1 + be^{-ay^\star}} = s_0 + s_1 y^\star + s_2 r^\star \tag{16}$$

$$\ell_0 + \ell_1 y^\star - \ell_2 r^\star = m^s - \overline{p}^\star \tag{17}$$

$$\overline{p}^\star = -\ln d_1 def = -\overline{d}_1 \tag{18}$$

In virtue of (18) from (16) and (17) the equilibrium values $y^\star$ and $r^\star$ can be found as a solution to

$$\frac{k}{1 + be^{-ay^\star}} = (s_0 + s_1 y^\star + s_2 r^\star)(1 + r^\star) \tag{19}$$

$$r^\star = \frac{1}{\ell_2}(\ell_0 - (m^s + \overline{d}_1) + \ell_1 y^\star) \Longleftrightarrow y^\star = \frac{1}{\ell_1}\left((m^s + \overline{d}_1) - \ell_0 + \ell_2 r^\star\right) \tag{20}$$

From (19) and (20) we get

$$\left[s_0 + s_2\left(\frac{\ell_0}{\ell_2} - \frac{m^s + \overline{d}_1}{\ell_2}\right) + \left(s_1 + s_2 \frac{\ell_1}{\ell_2}\right) y^\star\right] \cdot \left[1 + \frac{\ell_0}{\ell_2} - \frac{m^s + \overline{d}_1}{\ell_2} + \frac{\ell_1}{\ell_2} y^\star\right] = k \cdot \frac{1}{1 + be^{-ay^\star}} \tag{21}$$

Hence finding the solution to (21) and inserting this value into (20) we immediately get the pair of equilibrium points $y^\star, r^\star$. We can observe that:

The RHS of (21) is the so-called logistic function – an increasing function having an inflection point at $y = \frac{1}{a}\ln b$ which is convex in the interval $(0, \frac{1}{a}\ln b)$ and concave in $\left(\frac{1}{a}\ln b, \infty\right)$;

The LHS of (21) is a quadratic function (in fact, for real-life models this function differs only slightly from a straight line).

Hence there exist at most three, and in real models usually only one, pair(s) of equilibrium points $y^\star$, $r^\star$ for $y \geq 0$. More insight into the properties of the equilibrium points, especially with respect to stability, can be obtained by linearization around the neighborhood of the equilibrium point $(y^\star, r^\star, \pi^{e\star}, \overline{p}^\star)$ with $\pi^{e\star} = 0$. To check the stability of the linearized model (i.e., that all eigenvalues of the matrix of the linearized system have negative real parts), let us recall that all eigenvalues of the matrix lay in the union of the Gershgorin circles. The centers of the circles are diagonal elements of the matrix and the radius is equal to the minimum of the row or column sums of the absolute values of the corresponding off-diagonal elements. For details see e.g. (Fiedler, 1981).

3. Stability and Speed of Adjustment

3.1. Keynesian Model

In particular, on employing (16), (17), and (18) for the Keynesian model we have:

$$\begin{bmatrix} \frac{d(y(t)-y^\star)}{dt} \\ \frac{d(r(t)-r^\star)}{dt} \\ \frac{d(\pi^e(t))}{dt} \\ \frac{d(p(t)-p^\star)}{dt} \end{bmatrix} = \begin{bmatrix} \alpha(D_y - s_1) & \alpha(D_r - s_2) & 0 & 0 \\ \beta\ell_1 & -\beta\ell_2 & -\beta(\ell_2+\ell_3) & \beta \\ 0 & 0 & -\gamma & \gamma d_1 \\ 0 & 0 & 0 & -\delta d_1 \end{bmatrix} \begin{bmatrix} y(t)-y^\star \\ r(t)-r^\star \\ \pi^e(t) \\ p(t)-p^\star \end{bmatrix} \qquad (22)$$

where

$$D_y = \left.\frac{1}{1+r}\right|_{r=r^\star} \cdot \left.\frac{\partial}{\partial y}\frac{k}{1+be^{-ay(t)}}\right|_{y=y^\star}, \quad D_r = \left.\frac{\partial}{\partial r}\frac{1}{1+r}\right|_{r=r^\star} \cdot \left.\frac{k}{1+be^{-ay(t)}}\right|_{y=y^\star}$$

and

$$k = \left[s_0 + s_2\left(\frac{\ell_0}{\ell_2} - \frac{m^s + \bar{d}_1}{\ell_2}\right) + \left(s_1 + s_2\frac{\ell_1}{\ell_2}\right)y^\star\right] \cdot \left[1 + \frac{\ell_0}{\ell_2} - \frac{m^s + \bar{d}_1}{d_1\ell_2} + \frac{\ell_1}{\ell_2}y^\star\right] \cdot \left[1 + be^{-ay^\star}\right]$$

To verify if the obtained equilibrium point is stable, we shall have a look at the eigenvalues of the matrix

$$\boldsymbol{A} = \begin{bmatrix} \alpha(D_y - s_1) & \alpha(D_r - s_2) & 0 & 0 \\ \beta\ell_1 & -\beta\ell_2 & -\beta(\ell_2+\ell_3) & \beta \\ 0 & 0 & -\gamma & \gamma d_1 \\ 0 & 0 & 0 & -\delta d_1 \end{bmatrix} \qquad (23)$$

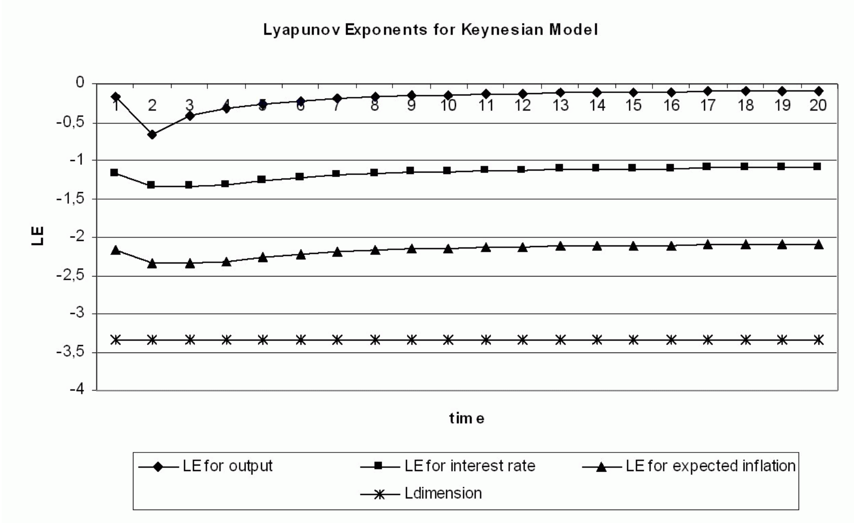
Employing the "nearly" upper triangular structure of the matrix $\boldsymbol{A}$ we can immediately conclude that the eigenvalues λ_1, λ_2, λ_3, λ_4 of $\boldsymbol{A}$ are equal to $\delta d_1, \gamma$ and the remaining two eigenvalues λ_3, λ_4 can be calculated as the two eigenvalues of the matrix

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \alpha(D_y - s_1) & \alpha(D_r - s_2) \\ \beta\ell_1 & -\beta\ell_2 \end{bmatrix} \qquad (24)$$

In particular, if the following two equations of the Keynesian model

$$\begin{bmatrix} \frac{d(y(t)-y^\star)}{dt} \\ \frac{d(r(t)-r^\star)}{dt} \end{bmatrix} = \begin{bmatrix} \alpha(D_y - s_1) & \alpha(D_r - s_2) \\ \beta\ell_1 & -\beta\ell_2 \end{bmatrix} \begin{bmatrix} y(t)-y^\star \\ r(t)-r^\star \end{bmatrix} \qquad (25)$$

FIGURE 1



are stable, then our extended Keynesian model given by (22) must also be stable. Obviously, the eigenvalues of $\tilde{A}$ are as follows (the symbols tr $\tilde{A}$ and det $\tilde{A}$ are reserved for the trace and determinant of $\tilde{A}$)

$$\lambda_{3,4} = \frac{1}{2}\left(tr\,\tilde{A} \pm \sqrt{(tr\,\tilde{A})^2 - 4\,det\,\tilde{A}} \right)$$

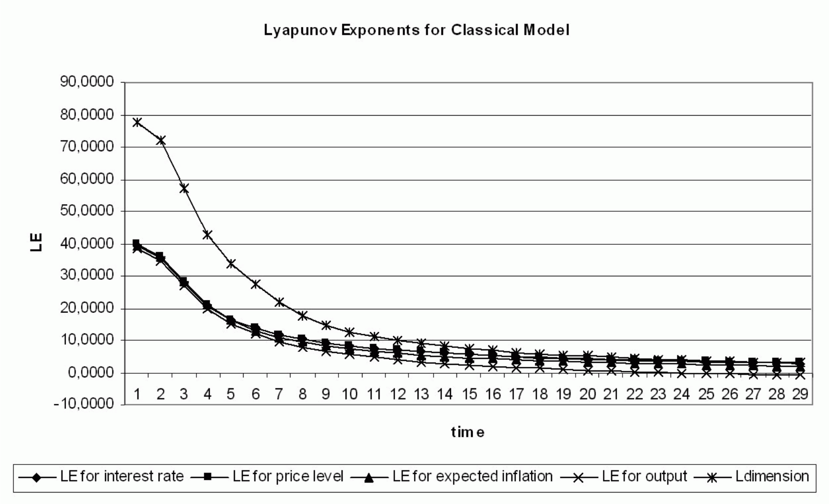
and det $\tilde{A}$ must be positive in order to exclude the possibility of a saddle point. For asymptotic stability $\operatorname{Re}\lambda_{3,4} < 0$, hence if $tr\,\tilde{A} = \alpha(D_y - s_1) - \beta\ell_2 < 0$ both (23) and (24) are stable, and if $\alpha(D_y - s_1) > \beta\ell_2$ the equilibrium is not asymptotically stable and a limit cycle occurs. In particular, the sufficient conditions for stability of matrix $\boldsymbol{A}$ of the considered four-equation Keynesian model are $D_y - s_1 < 0$ along with $D_y - s_1 > D_r - s_2$, $\ell_1 < \ell_2$ or $D_y - s_1 > \frac{\beta}{\alpha}\cdot\ell_1$, $D_r - s_2 > \frac{\beta}{\alpha}\cdot\ell_2$. An interesting case is when the eigenvalues of $\tilde{A}$ are purely imaginary, i.e., if $\alpha(D_y - s_1) = \beta\ell_0$.

The Lyapunov exponents for the considered four-equation Keynesian model with the following parameter values

$$\alpha = 20,\ \beta = 1,\ \gamma = 0.1,\ \delta = 0.02,\ a = 0.1,\ b = 1.5,\ s_0 = -0.16, s_1 = 0.07,\ s_2 = 0.016$$
$$l_0 = 0.25,\ l_1 = 0.4,\ l_2 = -0.06,\ l_3 = -0.06,\ d = -1,\ d_2 = 0.3, m^s = 0.65,\ k = 0.4$$

are presented in *Figure 1*. The Lyapunov dimension of the Keynesian model attractor is equal to 0. This means that the real parts of all the eigenvalues of the Keynesian model attractor are negative. Thus the Keynesian model is not a chaotic macroeconomic system.

FIGURE 2



3.2 Classical Model

In particular, on employing (16), (17), and (18) for the Classical model we have:

$$\begin{bmatrix} \dfrac{d\left(r(t)-r^{\star}\right)}{dt} \\ \dfrac{d\left(p(t)-p^{\star}\right)}{dt} \\ \dfrac{d\left(\pi^{e}(t)\right)}{dt} \end{bmatrix} = \begin{bmatrix} \alpha\left(D_r - s_2\right) & 0 & 0 \\ \beta\ell_2 & -\beta & \beta\left(\ell_2+\ell_3\right) \\ 0 & \gamma d_1 & -\gamma \end{bmatrix} \begin{bmatrix} r(t)-r^{\star} \\ p(t)-p^{\star} \\ \pi^{e}(t) \end{bmatrix} \tag{26}$$

where D_r and k take on the same values as in Section 3.1.

The Lyapunov exponents for the classical model with the following parameter values

$\alpha = 200,\ \beta = 0.2,\ \gamma = \delta = 1,\ a = 0.1,\ b = 1.5,\ s_0 = -0.16,\ s_1 = 0.07,\ s_2 = 0.016$

$l_0 = 0.25,\ l_1 = 0.4,\ l_2 = -0.06,\ l_3 = -0.06,\ d = -1,\ d_2 = 0.3,\ m^s = 0.65,\ k = 0.4,\ y = 4.5$

are presented in *Figure 2*. It shows that one of the Lyapunov exponents for the classical model attractor is equal to 0. This means that one real part of the eigenvalues is zero and the other real parts of the eigenvalues of the classical model attractor are negative. The Lyapunov dimension for the classical model attractor is also equal to 0. Thus the classical model can exhibit a limit cycle.

4. Conclusions

Macroeconomic models – the Keynesian model and the classical model – were analyzed with respect to both their stability and their speed of adjustment. Using different analysis methods (eigenvalues and Lyapunov exponents), it was shown that the Keynesian model is not a chaotic macroeconomic system. On the contrary, it was shown that the classical model can exhibit a limit cycle.

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