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On Some Properties of Cost Allocation Rules in Minimum Cost Spanning Tree Problems

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Abstract We consider four cost allocation rules in minimum cost spanning tree problems. These rules were introduced by Bird (1976), Dutta and Kar (2004), Kar (2002), and Feltkamp, Tijs and Muto (1994), respectively. We give a list of desirable properties and we study which properties are satisfied by these rules.

Keywords Minimum cost spanning tree, properties
JEL classification C71

1. Introduction

In this paper we study minimum cost spanning tree problems $(mcstp)$. Consider that a group of agents, located at different geographical places, want some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source.

There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence have to share cost of the distribution network. This example appears in Dutta and Kar (2004).

Bergantiños and Lorenzo (2004, 2005) studied a real situation where villagers should pay the cost of constructing pipes from their respective houses to a water supplier. Some houses in a valley sited in Galicia (Spain), required access to a water dam built by the local authority. The cost of the pipes connected to the houses water supply required villagers to pay for it. Some villagers paid for the pipes which would connect them to the dam. After the commencement of the system, most of the other villagers decided to connect to the network. This becomes a source of dispute as the latter villagers want usage of the existing network, paid by the former villagers creating a discompensation to former villagers. This situation could have been avoided by using a prior sharing cost rule.

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The literature on *mcstp* starts by defining algorithms for constructing minimum cost spanning trees (*mt*). Other important issue is how to allocate the cost associated with the *mt* among the agents.

Bird (1976) proposed a cost allocation rule (we call it B). B has been axiomatically characterized in Dutta and Kar (2004), Gómez-Rúa and Vidal-Puga (2005) and Özsoy (2006). Bird (1976) associated a coalitional game with any *mcstp*. Kar (2002) characterized the Shapley value of this coalitional game as an allocation rule for *mcstp*. We denote this rule as K. Dutta and Kar (2004) proposed and characterized a new rule, which we denote as *DK*. Finally, Feltkamp, Tijs and Muto (1994) introduced a rule for *mcstp*, we call it FTM. This rule has been axiomatically characterized in Brânzei, Moretti, Norde and Tijs (2004), and in Bergantiños and Vidal-Puga (2005, 2007a, 2007b).

In Bergantiños and Vidal-Puga (2007a), we gave a list of desirable properties that a fair rule should satisfy. Most of these properties are already known in the literature of *mcstp*, others are defined applying well-known principles to *mcstp*. We proved that FTM satisfies most of these properties. In this paper we study which of these properties are satisfied by the above rules.

In Bergantiños and Vidal-Puga (2007a), we defined the property of *Independence of Other Costs* (*IOC*). This property says that the amount paid by agent i depends only on the cost of the arcs to which he belongs. However, no rule satisfies *IOC*. In this paper we introduce two weaker versions of *IOC*. *Independence of Small Costs* (*ISC*) says that the amount paid by each agent i does not depend on the cost of the arcs that are cheaper than agent i's cheapest arc. *Independence of Large Costs* (*ILC*) says that the amount paid by each agent i does not depend on the cost of the arcs that are more expensive than agent i's most expensive arc. We prove that B satisfies *ISC* but fails *ILC*. K satisfies *ISC* but fails *ILC*. *DK* fails both. Nevertheless, FTM satisfies both.

The paper is organized as follows. In Section 2 we introduce *mcstp*, along with the rules and properties considered in the paper. In Section 3 we present the results. In Section 4 we provide some concluding remarks.

2. The minimum cost spanning tree problem

This section is divided in three subsections. In the first subsection, we introduce the problem. In the second subsection, we introduce the three rules of the literature. Finally, in the third subsection, we present the properties.

2.1 The problem

Let $\mathcal{N} = \{1,2,...\}$ be the set of all possible agents. Let Π_N be the set of all permutations over the finite set $N \subset \mathcal{N}$. Given $\pi \in \Pi_N$, let $Pre(i,\pi)$ denote the set of elements of N which come before i in the order given by π, i.e. $Pre(i,\pi) = \{j \in N \mid \pi(j) < \pi(i)\}$. Given $S \subset N$, let π_S denote the order induced by π among the agents in S.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1,...,n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we work with undirected arcs, i.e. $(i, j) = (j, i)$. We denote the set of all cost matrices over N as $\mathscr{C}^N$. Given $C, C' \in \mathscr{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly an *mcstp*, is a pair (N_0, C) where $N \subset \mathscr{N}$ is the finite set of agents, 0 is the source, and $C \in \mathscr{C}^N$ is the cost matrix. Given an *mcstp* (N_0, C), we define the *mcstp* induced by C in $S \subset N$ as (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) \,|\, i, j \in N_0\}$. The elements of g are called *arcs*. Given a network g and a pair of nodes i and j, a *path* from i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$, and $j = i_l$.

A *tree* is a network satisfying that for each $i \in N$ there exits a unique path from i to the source. If t is a tree we usually write $t = \left\{\left(i^0, i\right)\right\}_{i \in N}$, where i^0 represents the first node in the unique path in t from i to 0.

Let $\mathscr{G}^N$ denote the set of all networks over N_0. Let $\mathscr{G}_0^N$ denote the set of all networks, where each agent $i \in N$ is connected to the source, i.e. there exists a path from i to 0 in the network. Given an *mcstp* (N_0, C) and $g \in \mathscr{G}^N$, we define the *cost* associated with g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$. A *minimum cost spanning tree* for (N_0, C), briefly an *mt*, is a tree $t \in \mathscr{G}_0^N$ such that $c(t) = \min_{g \in \mathscr{G}_0^N} c(g)$. It is well-known that an *mt* exists, even though it does not necessarily have to be unique. Given an *mcstp* (N_0, C), we denote the cost associated with any *mt* t in (N_0, C) as $m(N_0, C)$.

Given an *mcstp*, Prim (1957) provided an algorithm for building an *mt*. The idea of this algorithm is simple: starting from the source we construct a network by sequentially adding arcs with the lowest cost and without introducing cycles. Formally, Prim's algorithm is defined as follows. We start with $S^0 = \{0\}$ and $g^0 = \emptyset$.

Stage 1 : Take an arc $(0, i)$ such that $c_{0i} = \min_{j \in N} \left\{c_{0j}\right\}$. If there are several arcs satisfying this condition, select one of them. Now, $S^1 = \{0, i\}$ and $g^1 = \{(0, i)\}$.

Stage $p+1$**:** Assume that we have defined $S^p \subset N_0$ and $g^p \in \mathscr{G}^N$. We now define S^{p+1} and g^{p+1}. Take an arc (j, i) with $j \in S^p$ and $i \in N_0 \backslash S^p$ such that $c_{ji} = \min_{k \in S^p, l \in N_0 \backslash S^p} \{c_{kl}\}$. If there are several arcs satisfying this condition, select one of them. Now, $S^{p+1} = S^p \cup \{i\}$ and $g^{p+1} = g^p \cup \{(j, i)\}$.

This process is completed in n stages. We say that g^n is a tree obtained following Prim's algorithm. Notice that this algorithm leads to a tree, but that this is not always unique. We use Prim's algorithm to prove the following result.

Lemma 1. *A tree t is an mt if and only if for all $S \subsetneq N_0$, $S \neq \emptyset$, there exists $(i, j) \in t$ with $i \in S$, $j \in N_0 \backslash S$ such that $c_{ij} = \min_{k \in S, l \in N_0 \backslash S} \{c_{kl}\}$.*

Proof. $(\Rightarrow)$ Let $S \subsetneq N_0$ such that $S \neq \emptyset$. Let (i,j) be an arc with $i \in S$, $j \in N_0 \backslash S$ and $c_{ij} = \min_{k \in S, l \in N_0 \backslash S} \{c_{kl}\}$. Suppose $(i,j) \notin t$. We consider the graph $g = t \cup \{(i,j)\}$. Since t is a tree, there exists a path in t from i to j. In this path, there exists an arc $(i',j') \in t$ with $(i,j) \neq (i',j')$, $i' \in S$ and $j' \in N_0 \backslash S$. Thus, $t' = g \backslash \{(i',j')\}$ is a tree in (N_0, C). Since t is an *mt*, $c_{i'j'} \leq c_{ij}$. Hence, $c_{i'j'} = \min_{k \in S, l \in N_0 \backslash S} \{c_{kl}\}$.
$(\Leftarrow)$ We prove that t is an *mt* proving that it can be obtained through Prim's algorithm. There exists $(0,i_1) \in t$ with $i_1 \in N$ such that $c_{0i_1} = \min_{j \in N} \{c_{0j}\}$. Thus, $(0,i_1)$ is an eligible arc in the first step of Prim's algorithm and $S^1 = \{0, i_1\}$. There exists $\left(i_2^0, i_2\right) \in t$ with $i_2^0 \in S^1$ and $i_2 \in N \backslash \{i_1\}$ such that $c_{i_2^0 i_2} = \min_{k \in S^1, l \in N_0 \backslash S^1} \{c_{kl}\}$. Thus, $\left(i_2^0, i_2\right)$ is again eligible following Prim's algorithm. Following this reasoning, we deduce that t can be computed following Prim's algorithm. Hence, t is an *mt*. □

A *game with transferable utility*, *TU game*, is a pair (N,v) where $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. $Sh(N,v)$ denotes the Shapley value (Shapley, 1953) of (N,v). Bird (1976) associated a *TU* game (N, v_C) with each *mcstp* (N_0, C). For each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

This is a "pessimistic" approach, because the players in S assume that the rest of the players are not present. An alternative approach is to assume that the rest of the players are already connected and thus connection is possible through them. In Bergantiños and Vidal-Puga (2007b), we associated an "optimistic" *TU* game $\left(N, v_C^+\right)$ with each *mcstp* (N_0, C). For each coalition $S \subset N$,

$$v_C^+(S) = m\left(S_0, C^{+(N \backslash S)}\right),$$

where $c_{ij}^{+(N \backslash S)} = c_{ij}$ for all $i, j \in S$ and $c_{i0}^{+(N \backslash S)} = \min_{j \in N_0 \backslash S} \{c_{ij}\}$ for all $i \in S$.

2.2 Rules

One of the most important issues addressed in the literature about *mcstp* is how to divide the cost of connecting agents to the source. A *cost allocation rule* is a function ψ such that $\psi(N_0, C) \in \mathbb{R}^N$ for each *mcstp* (N_0, C) and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. As usual, $\psi_i(N_0, C)$ represents the cost allocated to agent i.

We will now introduce four rules. The *Bird rule* (Bird 1976) and Dutta-Kar's rule (Dutta and Kar 2004) are defined through Prim's algorithm. We first assume that there is a unique *mt* t.

Given $i \in N$, let i^0 be the first node in the unique path in t from i to the source. The Bird rule (B) is defined for each $i \in N$ as

$$B_i(N_0, C) = c_{i^0 i}.$$

The idea of this rule is simple. Agents connect sequentially to the source following Prim's algorithm and each agent pays the corresponding connection cost.

Dutta-Kar's rule (DK) is defined in a more elaborate way. Assume that the agents, according with Prim's algorithm, connect in the order $1, 2, \ldots, n$. First agent 1 connects to the source. We define $p^1 = c_{01}$. Now agent 2 connects to 2^0 where $c_{2^0 2} =$

$min\{c_{02}, c_{12}\}$. We take $x_1 = min\{p^1, c_{2^0 2}\}$ and $p^2 = max\{p^1, c_{2^0 2}\}$. Now agent 3 connects to 3^0 where $c_{3^0 3} = min\{c_{03}, c_{13}, c_{23}\}$. We take $x_2 = min\{p^2, c_{3^0 3}\}$ and $p^3 = max\{p^2, c_{3^0 3}\}$. This process continue until we reach agent n. In this case we take $x_n = max\{p^{n-1}, c_{n^0 n}\}$. Then, the final allocation is given by x, i.e., for all $i \in N$

$$DK_i(N_0, C) = x_i.$$

Assume now there exists more than one *mt*. In this case, the Bird rule and Dutta-Kar's rule can be defined as an average of the trees associated with Prim's algorithm. Dutta and Kar (2004) proceeded as follows. Given $\pi \in \Pi_N$ they defined $B^\pi(N_0, C)$ as the allocation obtained when they applied the previous protocol to (N_0, C) and solved the indifferences by selecting the first agent given by π. Then they defined

$$B(N_0, C) = \frac{1}{n!} \sum_{\pi \in \Pi_N} B^\pi(N_0, C).$$

They defined $DK(N_0, C)$ in a similar way.

The game theory approach can also be used for defining rules. The *Kar rule* (K) is defined as

$$K(N_0, C) = Sh(N, v_C).$$

In Bergantiños and Vidal-Puga (2007b), we proved that *Feltkamp-Tijs-Muto's rule* (FTM) can be defined as

$$FTM(N_0, C) = Sh\left(N, v_C^+\right).$$

2.3 Properties

We now introduce several properties of rules. For a detailed discussion of these properties see, for instance, Bergantiños and Vidal-Puga (2007a). Given a rule ψ, we consider the following properties:

Core Selection (CS) For all *mcstp* (N_0, C) and all $S \subset N$, we have

$$\sum_{i \in S} \psi_i(N_0, C) \leq m(S_0, C).$$

CS says that no group of agents can be better off by building their own network. In particular, this property prevents some agents to subsidy others.

Cost Monotonicity (CM) For all *mcstp* (N_0, C) and (N_0, C') such that $c_{ij} < c'_{ij}$ for some $i \in N$, $j \in N_0$ and otherwise $c_{kl} = c'_{kl}$, we have

$$\psi_i(N_0, C) \leq \psi_i\left(N_0, C'\right).$$

CM says that a decrease in the cost of a link cannot harm their adjacent agents. In particular, this property prevents the agents to take advantage by reporting false connection costs.

Strong Cost Monotonicity (*SCM*)] For all *mcstp* (N_0, C) and (N_0, C') such that $C \leq C'$, we have

$$\psi(N_0, C) \leq \psi\left(N_0, C'\right).$$

SCM says that a decrease in the cost of a link cannot harm any agent. *SCM* is called solidarity in Bergantiños and Vidal-Puga (2007a).

Population Monotonicity (*PM*) For all *mcstp* (N_0, C), $S \subset N$, and $i \in S$, we have

$$\psi_i(N_0, C) \leq \psi_i(S_0, C).$$

PM says that no agent is worse off with the entrance of new agents. In particular, this property prevents incentives to veto the entrance of new agents.

Continuity (*CON*) For all $N \subset \mathcal{N}$, $\psi(N_0, \cdot)$ is a continuous function of $\mathcal{C}^N$. *CON* says that small changes in the costs do not mean a big change in the allocation.

Positivity (*POS*) For all *mcstp* (N_0, C) and all $i \in N$, we have

$$\psi_i(N_0, C) \geq 0.$$

POS says that no agent can make a profit.

Separability (*SEP*) For all *mcstp* (N_0, C) and $S \subset N$ satisfying $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$, we have

$$\psi_i(N_0, C) = \begin{cases} \psi_i(S_0, C) & \text{if } i \in S \\ \psi_i((N \setminus S)_0, C) & \text{if } i \in N \setminus S. \end{cases}$$

SEP says that if two groups of agents can connect to the source independently, then their respective allocations should also be independent.

Symmetry (*SYM*) For all *mcstp* (N_0, C) and all pair of symmetric agents $i, j \in N$,

$$\psi_i(N_0, C) = \psi_j(N_0, C).$$

We say that $i, j \in N$ are *symmetric* if for all $k \in N_0 \setminus \{i, j\}$, $c_{ik} = c_{jk}$.

Independence of Other Costs (*IOC*) For all *mcstp* (N_0, C) and (N_0, C'), and all $i \in N$ such that $c_{ij} = c'_{ij}$ for all $j \in N_0 \setminus \{i\}$, we have

$$\psi_i(N_0, C) = \psi_i\left(N_0, C'\right).$$

IOC says that an agent's allocation should only depend on the cost of their adjacent links.

Equal Share of Extra Costs (*ESEC*) Let (N_0, C) and (N_0, C') be two *mcstp*. Let $c_0, c'_0 \geq 0$. Assuming $c_{0i} = c_0$ and $c'_{0i} = c'_0$ for all $i \in N$, $c_0 < c'_0$, and $c_{ij} = c'_{ij} \leq c_0$ for all $i, j \in N$, we have

$$\psi_i\left(N_0, C'\right) = \psi_i(N_0, C) + \frac{c'_0 - c_0}{n}.$$

ESEC says that the agents should share equally any extra cost of direct connection to the source, when it is the more expensive one and it is equal for all the agents. We say that two *mcstp* (N_0,C) and (N_0,C') are *tree-equivalent* if there exists a tree t such that, firstly, t is an *mt* for both (N_0,C) and (N_0,C'), and secondly, $c_{ij}=c'_{ij}$ for all $(i,j)\in t$.

Independence of Irrelevant Trees (IIT) If two *mcstp* (N_0,C) and (N_0,C') are tree-equivalent,

$$\psi(N_0,C)=\psi\left(N_0,C'\right).$$

IIT says that any *mt* provides all the relevant information.

In Bergantiños and Vidal-Puga (2007a), we proved that there is no rule satisfying IOC. We now introduce two properties weaker than IOC.

Independence of Small Costs (ISC) Let (N_0,C) and (N_0,C') be two *mcstp* and $i\in N$ satisfying three conditions: First, $c_{ik}=c'_{ik}$ for all $k\in N_0$. Second, given $j,k\in N_0$, then $c_{jk}\le c_i^{min}$ if and only if $c'_{jk}\le c_i^{min}$ where $c_i^{min}=\min_{k\in N_0\setminus\{i\}}\{c_{ik}\}$. Third, given $j,k\in N_0$ such that $c_i^{min}<c_{jk}$ then, $c'_{jk}=c_{jk}$. Then,

$$\psi_i(N_0,C)=\psi_i\left(N_0,C'\right).$$

ISC says that the amount paid by agent i does not depend on the cost of the arcs cheaper than his cheapest arc.

Independence of Large Costs (ILC) Let (N_0,C) and (N_0,C') be two *mcstp* and $i\in N$ satisfying three conditions: First, $c_{ik}=c'_{ik}$ for all $k\in N_0$. Second, given $j,k\in N_0$, then $c_i^{max}\le c_{jk}$ if and only if $c_i^{max}\le c'_{jk}$ where $c_i^{max}=\max_{k\in N_0\setminus\{i\}}\{c_{ik}\}$. Third, given $j,k\in N_0$ such that $c_{jk}<c_i^{max}$ then, $c'_{jk}=c_{jk}$. Then,

$$\psi_i(N_0,C)=\psi_i\left(N_0,C'\right).$$

ILC says that the amount paid by agent i does not depend on the cost of the arcs larger than his most expensive arc.

In the next proposition we summarize the relations among these properties. Parts (i) and (ii) appear in Bergantiños and Vidal-Puga (2007a). Part (iii) is proved in this paper.

Proposition 1.

(i) *SCM implies CM and IIT.*
(ii) *PM implies CS and SEP.*
(iii) *IIT implies ILC.*

Proof. (iii) Assume that ψ is a rule satisfying IIT and let (N_0,C), (N_0,C') and $i\in N$ be as in the definition of ILC. We first prove the following claim:

Claim. If there exist $j,k\in N_0$ and $a>0$ such that, for all $l,m\in N_0$,

$$c_{lm}=\begin{cases} c'_{lm}-a & \text{if } (l,m)=(j,k)\\ c'_{lm} & \text{otherwise}\end{cases}$$

and $c_i^{max}\le c_{jk}$, then $\psi_i(N_0,C)=\psi_i(N_0,C')$.

Since $c_i^{max} \leq c_{jk} < c'_{jk}$, we deduce that $i \neq j$ and $i \neq k$. Let $t = \left\{\left(l^0, l\right)\right\}_{l \in N}$ be an *mt* in (N_0, C). If $(j,k) \notin t$, then t is also an *mt* in (N_0, C'). Since ψ satisfies *IIT*, $\psi_i(N_0, C) = \psi_i(N_0, C')$. If $(j,k) \in t$, we can assume, without loss of generality, that $j = k^0$. Two cases are possible:

1. Link (j,k) is not in the unique path in t from i to 0. We define $t^* = (t \setminus \{(j,k)\}) \cup \{(i,k)\}$. It is trivial to see that t^* is a tree satisfying that

$$c(N_0, C, t^*) - c(N_0, C, t) = c_{ik} - c_{jk}.$$

   Since $c_i^{max} \leq c_{jk}$ we deduce that $c(N_0, C, t^*) \leq c(N_0, C, t)$. Thus, t^* is an *mt* in both (N_0, C) and (N_0, C'). Since ψ satisfies *IIT*, $\psi_i(N_0, C) = \psi_i(N_0, C')$.

2. Link (j,k) is in the unique path in t from i to 0. We define $t^* = (t \setminus \{(j,k)\}) \cup \{(0,i)\}$. Using similar arguments to those used in the first case we can conclude that $\psi_i(N_0, C) = \psi_i(N_0, C')$.

This concludes the proof of the claim.

Let $A_i = \left\{\left(i_l^1, i_l^2\right)\right\}_{l=1}^{p}$ be the set of arcs satisfying that $c_{i_l^1 i_l^2} \neq c'_{i_l^1 i_l^2}$. We take $C^0 = C$. For all $l = 1, \ldots, p$ we define the *mcstp* $\left(N_0, C^l\right)$ where $c^l_{i_l^1 i_l^2} = c'_{i_l^1 i_l^2}$ and $c^l_{lm} = c^{l-1}_{lm}$ otherwise. For each $l = 1, \ldots, p$ we take $(j,k) = \left(i_l^1, i_l^2\right)$. Under the claim, $\psi_i\left(N_0, C^{l-1}\right) = \psi_i\left(N_0, C^l\right)$ for all $l = 1, .., p$. Since $C^0 = C$ and $C^p = C'$, ψ satisfies *ILC*. □

3. Properties of the rules

In this section we study which properties the rules satisfy. Some of the results are already known in the literature. In this case we only refer to the paper in which it is proved.

Theorem 1.

(i) *B satisfies CS, POS, SYM, ESEC, and ISC. B does not satisfy CM, SCM, PM, CON, SEP, IIT, and ILC.*

(ii) *K satisfies CM, CON, SYM, ESEC, and ISC. K does not satisfy CS, SCM, PM, POS, SEP, IIT, and ILC.*

(iii) *DK satisfies CS, CM, POS, and SYM. DK does not satisfy SCM, PM, CON, SEP, ESEC, IIT, ISC, and ILC.*

(iv) *FTM satisfies CS, CM, SCM, PM, CON, POS, SEP, SYM, ESEC, IIT, ISC, and ILC.*

Proof. (i)

B* satisfies *CS. See Bird (1976).

B* satisfies *POS. It is trivial.

B* satisfies *SYM. Let i, j be two symmetric agents in (N_0, C). Given $\pi \in \Pi_N$ we define $\pi^{ij} \in \Pi_N$ such that $\pi^{ij}(i) = \pi(j)$, $\pi^{ij}(j) = \pi(i)$, and $\pi^{ij}(k) = \pi(k)$ for all $k \in N \setminus \{i, j\}$. It is trivial to see that $B_i^{\pi}(N_0, C) = B_j^{\pi^{ij}}(N_0, C)$. Thus,

$$\begin{aligned} B_i(N_0,C) &= \frac{1}{n!}\sum_{\pi\in\Pi_N} B_i^{\pi}(N_0,C) = \frac{1}{n!}\sum_{\pi\in\Pi_N} B_j^{\pi^{ij}}(N_0,C) \\ &= \frac{1}{n!}\sum_{\pi\in\Pi_N} B_j^{\pi}(N_0,C) = B_j(N_0,C). \end{aligned}$$

B* satisfies *ESEC. Let (N_0, C) and (N_0, C') be as in the definition of $ESEC$. It is straightforward to see that for all $\pi \in \Pi_N$,

$$B_i^{\pi}(N_0,C') = \begin{cases} B_i^{\pi}(N_0,C) + (c_0' - c_0) & \text{if } \pi(i) = 1 \\ B_i^{\pi}(N_0,C) & \text{otherwise.} \end{cases}$$

Now it is not difficult to check that B satisfies $ESEC$.

B* satisfies *ISC. Let (N_0, C), (N_0, C'), and $i \in N$ be as in the definition of *ISC*. It is enough to prove that $B_i^{\pi}(N_0, C) = B_i^{\pi}(N_0, C')$ for each $\pi \in \Pi_N$.

We first assume that there exits a unique arc (j, k) such that $c'_{jk} \neq c_{jk}$. This means $i \neq j$ and $i \neq k$. We can assume without loss of generality that $c'_{jk} < c_{jk} \leq c_i^{min}$. When we compute $B^{\pi}(N_0, C)$ (resp. $B^{\pi}(N_0, C')$) following Prim's algorithm, the agents connect sequentially to the source in a specific order. We denote this order as π^* (resp. $\pi^{*\prime}$). We consider three cases:

1. $\pi^*(i) < \min\{\pi^*(j), \pi^*(k)\}$. Thus, $Pre(i, \pi^*) = Pre(i, \pi^{*\prime})$. Hence,
$$B_i^{\pi}(N_0,C) = B_i^{\pi}(N_0,C') = \min_{l\in Pre(i,\pi^*)} \{c_{il}\}.$$

2. $\min\{\pi^*(j), \pi^*(k)\} < \pi^*(i) < \max\{\pi^*(j), \pi^*(k)\}$. We assume without loss of generality that $\pi^*(j) < \pi^*(i) < \pi^*(k)$. We know that $B_i^{\pi}(N_0, C) = c_{i^0 i}$. Thus, $c_i^{min} \leq c_{i^0 i} \leq c_{jk}$. Since $c_{jk} \leq c_i^{min}$ we have that $c_{jk} = c_{i^0 i}$ and $B_i^{\pi}(N_0, C) = c_i^{min}$. Since $c'_{jk} < c_{jk}$, $Pre(i, \pi^*) \subset Pre(i, \pi^{*\prime})$. Thus, $B_i^{\pi}(N_0, C') \leq B_i^{\pi}(N_0, C) = c_i^{min}$. Moreover, $B_i^{\pi}(N_0, C') = \min_{l\in Pre(i,\pi^{*\prime})}\{c_{il}\} \leq c_i^{min}$. Thus, $B_i^{\pi}(N_0, C') = c_i^{min}$.

3. $\max\{\pi^*(j), \pi^*(k)\} < \pi^*(i)$. Thus, $Pre(i, \pi^*) = Pre(i, \pi^{*\prime})$. Hence,
$$B_i^{\pi}(N_0,C) = B_i^{\pi}(N_0,C') = \min_{l\in Pre(i,\pi^*)} \{c_{il}\}.$$

Assume now that there are several arcs (j, k) such that $c'_{jk} \neq c_{jk}$. Let $A = \left\{\left(j_1^h, j_2^h\right)\right\}_{h=1}^{p}$ be the set of those arcs. We consider the family $\left\{\left(N_0, C^h\right)\right\}_{h=0}^{p}$ such that $C^0 = C$ and

$$c_{kl}^h = \begin{cases} c'_{j_1^h j_2^h} & \text{if } (k,l) = \left(j_1^h, j_2^h\right) \\ c_{kl}^{h-1} & \text{otherwise} \end{cases}$$

for all $h=1,...,p$. Notice that for all $h=1,...,p$, $\left(N_0,C^{h-1}\right)$, $\left(N_0,C^{h}\right)$, and $i\in N$ are as in the definition of ISC. Moreover, $\left(j_1^h,j_2^h\right)$ is the unique arc with different cost in both $mcst\,p$. Thus, $B_i^{\pi}\left(N_0,C^{h-1}\right)=B_i^{\pi}\left(N_0,C^{h}\right)$ for all $h=1,...,p$. Since $C^p=C'$, $B_i^{\pi}\left(N_0,C\right)=B_i^{\pi}\left(N_0,C'\right)$.

B* does not satisfy *CM. See Dutta and Kar (2004).

B* does not satisfy *SCM. Since B does not satisfy CM, under Proposition 1 (i), the result holds.

B* does not satisfy *CON.
Example 1. Let $\left(N_0,C^x\right)$ be such that $N=\{1,2\}$, $x\geq 0$, and

$$C^x=\begin{pmatrix} 0 & 10 & 10+x \\ 10 & 0 & 2 \\ 10+x & 2 & 0 \end{pmatrix}.$$

$B\left(N_0,C^x\right)=(10,2)$ when $x>0$ but $B\left(N_0,C^0\right)=(6,6)$.

B* does not satisfy *SEP.
Example 2. Let $\left(N_0,C\right)$ be such that $N=\{1,2,3\}$, $S=\{1,2\}$, and

$$C=\begin{pmatrix} 0 & 3 & 10 & 1 \\ 3 & 0 & 1 & 10 \\ 10 & 1 & 0 & 3 \\ 1 & 10 & 3 & 0 \end{pmatrix}.$$

It is clear that $m\left(S_0,C\right)+m\left(\left(N\backslash S\right)_0,C\right)=m\left(N_0,C\right)$. However, $B\left(N_0,C\right)=(2,2,1)$ and $B\left(S_0,C\right)=(3,1)$.

B* does not satisfy *PM. Since B does not satisfy SEP, under Proposition 1 (ii), the result holds.

***B* does not satisfy *ILC*.** In Example 1, $\left(N_0,C^0\right)$, $\left(N_0,C^2\right)$ and 1 are as in the definition of ILC. Nevertheless, $B_1\left(N_0,C^0\right)=6$ and $B_1\left(N_0,C^2\right)=10$.

B* does not satisfy *IIT. Since B does not satisfy ILC, under Proposition 1 (iii), the result holds.

(ii)

K* satisfies *CM. See Dutta and Kar (2004).

K* satisfies *CON. Since $K\left(N_0,C\right)=Sh\left(N,v_C\right)$ and v_C is a continuous function on C, the result holds.
K* satisfies *SYM. It is trivial to see that if agents i and j are symmetric in $\left(N_0,C\right)$, they are symmetric in $\left(N,v_C\right)$. Since $K\left(N_0,C\right)=Sh\left(N,v_C\right)$ and the Shapley value is symmetric, the result holds.
K* satisfies *ESEC. Let $\left(N_0,C\right)$ and $\left(N_0,C'\right)$ be as in the definition of $ESEC$. It is easy to see that $v_{C'}\left(S\right)=v_C\left(S\right)+\left(c_0'-c_0\right)$ for all $S\subset N$. Thus,

$$\begin{aligned} K_i\left(N_0,C'\right) &= Sh_i\left(N,v_{C'}\right)=Sh_i\left(N,v_C\right)+\frac{c_0'-c_0}{n} \\ &= K_i\left(N_0,C\right)+\frac{c_0'-c_0}{n}. \end{aligned}$$

K* satisfies *ISC. Let $(N_0,C),(N_0,C')$ and $i\in N$ be as in the definition of *ISC*. We assume that there exists a unique arc (j,k) such that $c_{jk}\neq c'_{jk}$. The general case can be derived from this case using similar arguments to those used with B. We assume, without loss of generality, that $c'_{jk}<c_{jk}$.
Since $K(N_0,C)=Sh(N,v_C)$ and $v_C(S)=m(S_0,C)$ for all $S\subset N$, it is enough to prove that

$$m(S_0,C)-m(S_0\backslash\{i\},C)=m\left(S_0,C'\right)-m\left(S_0\backslash\{i\},C'\right),$$

when $i\in S$. We prove it only when $\{j,k\}\subset S$. The other cases are trivial.
Let $t=\left\{\left(l^0,l\right)\right\}_{l\in S}$ be an *mt* in (S_0,C). We define R as the set of agents who are adjacent to agent i and connect to the source (in t) through agent i. Namely,

$$R=\left\{l\in S|l^0=i\right\}.$$

We assume, without loss of generality, that $R=\{1,2,...,r\}$ ($R=\emptyset$ is possible).
Given $l\in R$, we define R^l as the set of agents in S who connect to the source in t through agent l. Namely, R^l is the set of agents $p\in S$ such that agent l is in the unique path in t from p to 0. We consider $l\in R^l$. Moreover, we define

$$R^0=(S_0\backslash\{i\})\setminus\bigcup_{l\in R}R^l.$$

For each $l\in R$, we define the *mcstp* $\left(\left(R^l\setminus\{l\}\right)_0,C^{+l}\right)$ as the *mcstp* that results form $\left(R^l,C\right)$ when agent l connects to the source and the rest of the agents can connect through him. Formally, for all $p\in R^l\setminus\{l\}$, $c^{+l}_{0p}=\min\left\{c_{lp},c_{0p}\right\}$ and $c^{+l}_{qp}=c_{qp}$ when $q\neq 0$.
It is not difficult to check that

$$m(S_0,C)=m\left(R^0,C\right)+\sum_{l\in R}m\left(\left(R^l\setminus\{l\}\right)_0,C^{+l}\right)+\sum_{l\in R}c_{il}+c_{i^0i}.$$

Let $p_0^*\in R^0$ and $p_1\in\bigcup_{l\in R}R^l$ be such that their connection cost is minimal. Namely,

$$c_{p_0^*p_1}=\min\left\{c_{qs}|q\in R^0\text{ and }s\in\bigcup_{l\in R}R^l\right\}.$$

We can assume, without loss of generality, that $p_1\in R^1$. Let $p_1^*\in R^0\cup R^1$ and $p_2\in\bigcup_{l\in R\setminus\{1\}}R^l$ be such that their connection cost is minimal, hence

$$c_{p_1^*p_2}=\min\left\{c_{qs}:q\in R^0\cup R^1\text{ and }s\in\bigcup_{l\in R\setminus\{1\}}R^l\right\}.$$

Again, we can assume, without loss of generality, that $p_2\in R^2$.

Following this procedure, we obtain $\left\{\left(p_{l-1}^{*},p_{l}\right)\right\}_{l\in R}$ such that $p_l \in R^l$ for all $l=1,..,r$. Under Lemma 1,

$$t^{*}=\left(t\setminus\left(\left\{\left(i^{0},i\right)\right\}\cup\bigcup_{l\in R}\left\{(i,l)\right\}\right)\right)\cup\left\{\left(p_{l-1}^{*},p_{l}\right)\right\}_{l\in R}$$

is an *mt* in $(S_0\setminus\{i\},C)$. Thus,

$$\begin{aligned} m(S_0,C)-m(S_0\setminus\{i\},C) &= c(t)-c(t^*) \\ &= \sum_{l\in R}c_{il}+c_{i^0 i}-\sum_{l\in R}c_{p_{l-1}^{*}p_l}. \end{aligned}$$

We will prove that this expression coincides with $m(S_0,C')-m(S_0\setminus\{i\},C')$. Recall $c_{il}=c'_{il}$ for all $l\in N_0$, $i\notin\{j,k\}$. We see two cases:

1. There exists $l\in\{0,1,...,r\}$ such that $\{j,k\}\subset R^l$. We consider three subcases:

   (a) $(j,k)\in t$. Since $c'_{jk}<c_{jk}$, t is an *mt* in (N_0,C'), t^* is an *mt* in $(S_0\setminus\{i\},C')$, and $(j,k)\in t^*$. Hence,

   $$\begin{aligned} m\left(S_0,C'\right)-m\left(S_0\setminus\{i\},C'\right) &= \sum_{l\in R}c'_{il}+c'_{i^0 i}-\sum_{l\in R}c'_{p_{l-1}^{*}p_l}. \\ &= \sum_{l\in R}c_{il}+c_{i^0 i}-\sum_{l\in R}c_{p_{l-1}^{*}p_l} \\ &= m(S_0,C)-m(S_0\setminus\{i\},C). \end{aligned}$$

   (b) $(j,k)\notin t$ and $\{j,k\}\subset R^l$ with $l\neq 0$. We consider the graph $t\cup\{(j,k)\}$. Since t is a tree, there exists a cycle g in (S_0,C). Moreover, $(i,l)\notin g$ because $\{j,k\}\subset R^l$, (i,l) is in the unique path connecting j with 0, and (i,l) is also in the unique path connecting k with 0. Under Lemma 1, by deleting the most expensive arc in g we get an *mt* t' in (S_0,C'). Notice that $\{(i,l)\}_{l\in R}\in t'$, $\left(i^0,i\right)\in t'$, and $\left\{\left(p_{l-1}^{*},p_l\right)\right\}_{l\in R}\in t'$. Using arguments similar to those used before with t and t^* we obtain that

   $$m\left(S_0,C'\right)-m\left(S_0\setminus\{i\},C'\right)=\sum_{l\in R}c'_{il}+c'_{i^0 i}-\sum_{l\in R}c'_{p_{l-1}^{*}p_l},$$

   which coincides with $m(S_0,C)-m(S_0\setminus\{i\},C)$.

   (c) $(j,k)\notin t$ and $\{j,k\}\subset R^0$. We consider the graph $t\cup\{(j,k)\}$. Since t is a tree, there exists a cycle g in (S_0,C). Using arguments similar to those used in the previous case we can prove that

   $$m\left(S_0,C'\right)-m\left(S_0\setminus\{i\},C'\right)=m(S_0,C)-m(S_0\setminus\{i\},C).$$

2. $j\in R^l, k\in R^q$ with $l\neq q$. Thus, $l\neq 0$ or $q\neq 0$. Assume, without loss of generality, that $l\neq 0$. Then,

$$\hat{t}=(t\setminus\{(i,l)\})\cup\{(j,k)\}$$

is a tree in (S_0,C) and $c(S_0,C,\hat{t}) = c(S_0,C,t) - c_{il} + c_{jk}$. Since t is an mt in (S_0,C) and $c_{jk} \leq c_i^{min}$, we conclude that $c_{jk} = c_{il} = c_i^{min}$ and $\hat{t}$ is an mt in (S_0,C). We can compute $\hat{R}$ and $\{\hat{R}^l\}_{l \in R \cup \{0\}}$ for $\hat{t}$ in the same way that we computed R and $\{R^l\}_{l \in R}$ for t. Now, there exists $l \in R \cup \{0\}$ such that $\{j,k\} \subset \hat{R}^l$ and we proceed as in Case 1.

K* does not satisfy *CS. See Dutta and Kar (2004).

K* does not satisfy *ILC. In Example 1, (N_0,C^0), (N_0,C^2) and 1 are as in the definition of *ILC*. Nevertheless, $K_1(N_0,C^0) = 6$ and $K_1(N_0,C^2) = 5$.

K* does not satisfy *IIT. Since K does not satisfy *ILC*, under Proposition 1 (iii), the result holds.

K* does not satisfy *SCM. Since K does not satisfy *ILC*, under Proposition 1 (i) and Proposition 1 (iii), the result holds.

K* does not satisfy *PM. Since K does not satisfy *CS*, under Proposition 1 (ii), the result holds.

K* does not satisfy *POS. In Example 1, $K_1(N_0,C^{20}) = -4$.

K* does not satisfy *SEP.
Example 3. Let (N_0,C) be such that $N = \{1,2,3\}$ and

$$C = \begin{pmatrix} 0 & 10 & 100 & 20 \\ 10 & 0 & 10 & 100 \\ 100 & 10 & 0 & 40 \\ 20 & 100 & 40 & 0 \end{pmatrix}.$$

Take $S = \{1,2\}$. Then, $m(N_0,C) = 40$, $m(S_0,C) = 20$, $m(\{3\}_0,C) = 20$, $K_1(S_0,C) = -35$, and $K_1(N_0,C) = -15$.

(iii)

DK* satisfies *CS* and *CM. See Dutta and Kar (2004).

DK* satisfies *POS. It is trivial.

DK* satisfies *SYM. Using arguments similar to those used when we proved that B satisfies SYM, we can prove that DK also satisfies SYM.

DK* does not satisfy *ILC. In Example 1, (N_0,C^0), (N_0,C^2), and 1 are as in the definition of *ILC*. Nevertheless, $DK_1(N_0,C^0) = 6$ and $DK_1(N_0,C^2) = 2$.

DK* does not satisfy *SCM. Since DK does not satisfy *ILC*, under Proposition 1 (i) and Proposition 1 (iii), the result holds.

DK* does not satisfy *CON. In Example 1, $DK(N_0,C^x) = (2,10)$ when $x > 0$ and $DK(N_0,C^0) = (6,6)$.

DK* does not satisfy *SEP. In Example 2, $DK(N_0,C) = (2,2,1)$ and $DK(S_0,C) = (1,3)$.

DK* does not satisfy *PM. Since DK does not satisfy *SEP*, under Proposition 1 (ii), the result holds.

DK* does not satisfy *ESEC.

Example 4. Let (N_0, C) and (N_0, C') be such that $N = \{1,2,3\}$,

$$C = \begin{pmatrix} 0 & 10 & 10 & 10 \\ 10 & 0 & 10 & 10 \\ 10 & 10 & 0 & 6 \\ 10 & 10 & 6 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 16 & 16 & 16 \\ 16 & 0 & 10 & 10 \\ 16 & 10 & 0 & 6 \\ 16 & 10 & 6 & 0 \end{pmatrix}.$$

We have $DK(N_0, C) = (10, 8, 8)$ and $DK(N_0, C') = (14, 9, 9)$.

DK* does not satisfy *IIT. Since *DK* does not satisfy *ILC*, under Proposition 1 (iii), the result holds.

DK* does not satisfy *ISC.
Example 5. Let (N_0, C) and (N_0, C') be such that $N = \{1,2,3\}$,

$$C = \begin{pmatrix} 0 & 100 & 110 & 120 \\ 100 & 0 & 6 & 6 \\ 110 & 6 & 0 & 10 \\ 120 & 6 & 10 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 100 & 110 & 120 \\ 100 & 0 & 3 & 6 \\ 110 & 3 & 0 & 10 \\ 120 & 6 & 10 & 0 \end{pmatrix}$$

Then, $DK_3(N_0, C) = 53$ and $DK_3(N_0, C') = 100$.

(iv)

FTM* satisfies *CS, CM, SCM, PM, CON, POS, SEP, SYM, ESEC,* and *IIT. See Bergantiños and Vidal-Puga (2007a).

FTM* satisfies *ISC. Before proving it we need some previous results, which can be found at Bergantiños and Vidal-Puga (2007a).
Given an *mcstp* (N_0, C) and an *mt* t, Bird (1976) defined the *minimal network* (N_0, C^t). In Bergantiños and Vidal-Puga (2007a), we defined the *irreducible form* of an *mcstp* (N_0, C) as the minimal network (N_0, C^*) associated with any *mt*.
An *mcstp* (N_0, C^*) is *irreducible* if and only if there exists a tree t in (N_0, C^*) that satisfies the following two conditions:

$(A1)$ t is lineal, i.e. $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ where $\pi_0 = 0$.

$(A2)$ Given $\pi_p, \pi_q \in N_0$ with $p < q$, $c^*_{\pi_p \pi_q} = \max_{s|p<s\leq q} \left\{ c^*_{\pi_{s-1}\pi_s} \right\}$.

Moreover, t is an *mt*. Given an *mcstp* (N_0, C), we say that the agents *connect to the source via t' in the order π following Prim's algorithm* if t' is obtained through Prim's algorithm and in stage p, the arc selected is $\left(\pi_p^0, \pi_p\right)$, for each p. We define C^{**} as follows: for all $\pi_p, \pi_q \in N_0$ with $p < q$,

$$c^{**}_{\pi_p \pi_q} = \max_{s|p<s\leq q} \left\{ c_{\pi_s^0 \pi_s} \right\}.$$

The *mcstp* (N_0, C^{**}) is the irreducible form of (N_0, C), i.e. $C^{**} = C^*$. Moreover, $t = \{(\pi_{s-1}, \pi_s)\}_{s=1}^n$ is an *mt* in (N_0, C^*) that satisfies $(A1)$ and $(A2)$.
Let (N_0, C), (N_0, C') and $i \in N$ be as in the definition of *ILC*. We assume that there exists a unique arc (j, k) such that $c_{jk} \neq c'_{jk}$. The general case can be derived from this

case using similar arguments to those used with B and K. We assume, without loss of generality, that $c'_{jk} < c_{jk}$. Since $c'_{jk} < c_{jk} \leq c_i^{min}$, we deduce that $i \neq j$ and $i \neq k$. We consider three cases:

1. There exists an *mt*, $t = \left\{\left(l^0, l\right)\right\}_{l \in N}$ in (N_0, C) such that $(j,k) \in t$.

   Assume, without loss of generality, that $j < k$, $k^0 = j$, and that the agents connect to the source (in C) via t in the order $\pi = (1, \ldots, n)$ following Prim's algorithm.

   Since t is an *mt* in (N_0, C) and $(j,k) \in t$, we have that t is an *mt* in (N_0, C'). Let $\pi' = (\pi'_1, \ldots, \pi'_n)$ such that the agents connect to the source (in C') via t in the order π' following Prim's algorithm.

   We can find π' such that for each $l = 1, \ldots, j$, we have $l = \pi'_l$ and $c^*_{(l-1)l} = c'^*_{(l-1)l} = c_{l^0 l}$.

   Let p be such that $k = \pi'_p$. Thus, $j < p \leq k$. Moreover, we can choose π' such that $l = \pi'_l$ for all $l = j+1, \ldots, p-1$ and $c^*_{(l-1)l} = c'^*_{(l-1)l} = c_{l^0 l} < c_i^{min}$ for all $l = p, \ldots, k$.

   Assume that we can find $m \in N$ such that $m > k$, $c^*_{(m-1)m} = c_{m^0 m} \geq c_i^{min}$, and $c^*_{(l-1)l} = c'^*_{(l-1)l} = c_{l^0 l} < c_i^{min}$ for all $l = k+1, \ldots, m-1$. Then, for all $l = m, \ldots, n$, $l = \pi'_l$ and $c^*_{(l-1)l} = c'^*_{(l-1)l} = c_{l^0 l}$. Since $c_{l^0 l} < c_i^{min}$ for all $l = p, \ldots, m-1$ we deduce that $\pi'_i = i$. Moreover, $i < j$ or $i \geq m$.

   If we can not find m as above, then $\pi'_i = i < j$. In this case we take $m = n+1$.

   We now prove that $c^*_{il} = c'^*_{il}$ for all $l \in N_0 \setminus \{i\}$. We assume that $i < j$ (the case $i \geq m$ is similar and we omit it). By $(A2)$, it is trivial to see that $c^*_{il} = c'^*_{il}$ when $l \leq j$. If $j < l \leq m-1$, then $c^*_{il} = c^*_{ij} = c'^*_{ij} = c'^*_{il}$. If $l \geq m$, $c^*_{il} = \max\left\{c^*_{ij}, c^*_{(m-1)l}\right\} = \max\left\{c'^*_{ij}, c'^*_{(m-1)l}\right\} = c'^*_{il}$.

   In Bergantiños and Vidal-Puga (2007a, Lemma 4.1(b)), we proved that FTM satisfies *IOC* in the class of irreducible matrices. Thus, $FTM_i\left(N_0, C^*\right) = FTM_i\left(N_0, C'^*\right)$. From Bergantiños and Vidal-Puga (2007a, Definition 3.1), it is straightforward to check that for all *mcstp* (N_0, C), $FTM\left(N_0, C\right) = FTM\left(N_0, C^*\right)$. Hence, $FTM_i\left(N_0, C\right) = FTM_i\left(N_0, C'\right)$.

2. For all *mt* t in (N_0, C), $(j,k) \notin t$. Moreover, for all *mt* t' in (N_0, C'), $(j,k) \notin t'$.

   Let t be an *mt* in (N_0, C). Thus, t is also an *mt* in (N_0, C'). Since FTM satisfies IIT, $FTM_i\left(N_0, C\right) = FTM_i\left(N_0, C'\right)$.

3. For all *mt* t in (N_0, C), $(j,k) \notin t$. Moreover, there exists an *mt* t' in (N_0, C') such that $(j,k) \in t'$.

   Clearly, $m\left(N_0, C\right) > m\left(N_0, C'\right)$. We define the *mcstp* (N_0, C'') where $c''_{jk} = c'_{jk} + m\left(N_0, C\right) - m\left(N_0, C'\right)$ and $c''_{lm} = c_{lm}$ otherwise. Notice that $C \geq C'' \geq C'$.

   It is trivial to see that if t is an *mt* in (N_0, C), then t is an *mt* in (N_0, C''). Since FTM satisfies IIT, $FTM_i\left(N_0, C\right) = FTM_i\left(N_0, C''\right)$.

By Case 1, $FTM_i(N_0, C'') = FTM_i(N_0, C')$.

***FTM* satisfies *ILC*.** Since FTM satisfies IIT, under Proposition 1 (iii), it holds. □

In the Table 1 we summarize the results obtained in Theorem 1.

Table 1. Rules and its properties as stated in Theorem 1

	B	K	DK	FTM
CS	✓	–	✓	✓
CM	–	✓	✓	✓
SCM	–	–	–	✓
PM	–	–	–	✓
CON	–	✓	–	✓
POS	✓	–	✓	✓
SEP	–	–	–	✓
SYM	✓	✓	✓	✓
$ESEC$	✓	✓	–	✓
IIT	–	–	–	✓
ISC	✓	✓	–	✓
ILC	–	–	–	✓

4. Concluding remarks

We have studied different properties that are defined in the literature of cost allocation in minimum cost spanning tree problems (*mcstp*). Most of these properties have been previously studied in the literature and applied to some rules. However, not all the properties had been checked for all the rules. In this paper we fill this gap. There are other properties that have been studied in the literature. We briefly comment three of them: consistency, additivity, and strategic merging.

The idea of *consistency* is the following: Some agents pay the allocation that some rule assigns to them, and connect to the source. The rest of the agents face the resulting *mcstp* and pay the allocation that the same rule assigns to them. Consistency states that the final allocation is the same as before. Two different properties of consistency are used to characterize *DK* defined (Dutta and Kar, 2004, Theorem 2) and *B* (Dutta and Kar, 2004, Theorem 3), respectively.

Additivity implies that the solution for the sum of two problems should be the sum of their respective solutions. This property is too strong and no rule satisfies it. A restricted version of additivity is used to characterize FTM in Brânzei, Moretti, Norde and Tijs (2004) and Bergantiños and Vidal-Puga (2005).

Strategic merging arises when a group of agents manipulates the allocation by merging and acting as a single node. It is of interest that no improvement be possible via strategic merging. Among the above rules, only *B* satisfies this property in an wide class of problems. Non-strategic merging is used to characterize *B* in Gómez-Rúa and Vidal-Puga (2005) and Özsoy (2006).

References

Bergantiños, G. and Lorenzo, L. (2004). A Non-cooperative Approach to the Cost Spanning Tree Problem. *Mathematical Methods of Operations Research*, 59, 393–403.

Bergantiños, G. and Lorenzo, L. (2005). Optimal Equilibria in the Non-cooperative Game Associated with Cost Spanning Tree Problems. *Annals of Operations Research*, 137, 101–115.

Bergantiños, G. and Vidal-Puga, J. (2005). Additivity in Cost Spanning Tree Problems. (forthcoming in *Journal of Mathematical Economics*, DOI: 10.1016/j.jmateco.008.03.003)

Bergantiños, G. and Vidal-Puga, J. (2007a). A Fair Rule in Minimum Cost Spanning Tree Problems. *Journal of Economic Theory*, 137(1), 326–352.

Bergantiños, G. and Vidal-Puga, J. (2007b). The Optimistic TU Game in Minimum Cost Spanning Tree Problems. *International Journal of Game Theory*, 36(2), 223–239.

Bird, C. G. (1976). On Cost Allocation for a Spanning Tree: A Game Theoretic Approach. *Networks*, 6, 335–350.

Branzei, R., Moretti, S., Norde H. and Tijs S. (2004). The P-value for Cost Sharing in Minimum Cost Spanning Tree Situations. *Theory and Decision*, 56, 47–61.

Dutta, B. and Kar, A. (2004). Cost Monotonicity, Consistency and Minimum Cost Spanning Tree Games. *Games and Economic Behavior*, 48(2), 223–248.

Feltkamp, V., Tijs, S. and Muto, S. (1994). On the Irreducible Core and the Equal Remaining Obligation Rule of Minimum Cost Extension Problems. Mimeo, Tilburg University.

Gómez-Rúa, M. and Vidal-Puga, J. (2005). Non-manipulability in Cost Spanning Tree Problems. Mimeo. Avaliable at http://webs.uvigo.es/vidalpuga.

Özsoy, H. (2006). A Characterization of Bird's Rule. Mimeo. Available at http://www.owlnet.rice.edu/~ ozsoy/index_jobmarketpaper.html.

Kar, A. (2002). Axiomatization of the Shapley Value on Minimum Cost Spanning Tree Games. *Games and Economic Behavior*, 38, 265–277.

Prim, R. C. (1957). Shortest Connection Networks and Some Generalizations. *Bell Systems Technology Journal*, 36, 1389–1401.

Shapley, L. (1953). A Value for n-person Games. In Kuhn, H. W. and Tucker A. W. (eds.), *Contributions to the Theory of Games II*. Princeton, Princeton University Press, 307–317.